The octuplet of the year 2024 and its relatives

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Abstract. *In this note, we address questions of decompositions into* 8 *prime numbers of the integer* 2024 *and its relatives. Our approach is essentially made up of numerical experiments.*

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1. Introduction

Number theory, or better Number science, is an area of mathematics that is approached either from the point of view of the professional mathematician (whether a teacher or researcher) or as recreational mathematics. This is particularly true for prime numbers (or primes). A sample of references taken from our libraries (professional or personal) shows the variety of productions on the subject: [1, 5, 6, 8, 9, 12, 13, 14].

Both of us authors, have made careers in university teaching and research in areas of mathematics and mechanics that have little (or nothing) to do with Number science. But we have always kept a taste and an interest in numbers, prime numbers in particular. Retirement is an opportunity to revisit these neglected areas, aided by the current availability of computing power on computers (even laptops).

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The starting point of our work is the number 2024, the number of the current year, and the (very) specific decomposition as a sum of primes it enjoys (see (1) below). Our contribution is essentially made of large-scale numerical calculations.

Each integer has its specificities, recalled and collected occasionally in compilations such as [10], or at the beginning of the years. 2024 is no exception to the rule, even if neighboring integers are richer in properties. This note works around an exceptional property of the integer 2024, which is that of being *the sum of* 8 *(strictly) consecutive*[3] *prime numbers*:

$$2024 = 233 + 239 + 241 + 251 + 257 + 263 + 269 + 271. \quad (1)$$

Before going further in the exploration of this type of properties, since we are located in the country of PIERRE (DE) FERMAT[4], let us offer a fantasy, still linked to the integers marking the years. Here it is. Among FERMAT's many brilliant contributions, we choose his "two square theorem (for prime numbers)". This is good, it applies to 2017 and 2029 (the only two cases between the years 2000 and 2050), and we are, in 2024 when we finalize these lines, between the two roughly in the middle. Here is what FERMAT's result says: *A prime number is the sum of two squares of integers if, and only if, it is congruent to 1 modulo 4*. In this case, the two integers appearing in the decomposition are unique, one is even and the other is odd. So, $2017 = (44)^2 + (9)^2$ and $2029 = (2)^2 + (45)^2$.

Let us return to decomposition (1) of 2024. The 8 prime numbers entering into the decomposition (1) constitute what is called a *free octuplet* (or, if one prefers, with 7 degrees of freedom, as opposed to an *arithmetic octuplet* with just 1 degree of freedom, its so-called common difference) ; this octuplet is determined by 2 characteristics, the ordered values of the successive inceases $p_i - p_1, i = 2, \ldots, 8$, that we denote

$$g_0 = (6, 8, 18, 24, 30, 36, 38), \quad (2)$$

and the fact that the prime numbers in the decomposition are strictly consecutive.

Recall that the spectacular theorem of B. GREEN and T. TAO ([7]) states that the (infinite) sequence of prime numbers contains arbitrary long

[3] "strictly consecutive" is a wording to emphasize and remember that the primes appear in ascending order but without forgetting any. *Stricto sensu*, "consecutive" would be enough.

[4] By the way, while talking about 2024, it is the year of the inauguration and official opening of the FERMAT museum in his birthplace Beaumont-de-Lomagne (60 km northwest of Toulouse).

arithmetic progressions. In other words, for every integer $n > 2$, there exist arithmetic progressions of primes with n terms. So, there indeed exist arithmetic octuplets (this the specific case with $n = 8$). Green & Tao theorem is essentially an existence theorem; we do not know, as a general rule, neither the starting point of the progression (with n terms) nor the common difference r.

In the work [3], the first author described a constructive approach to these n-uplets and exhibited several representatives; he also showed the essential role of common differences r of the form *primorials*[5] $p_n\#$, in particular $3\# = 6$. In a further work [4], admitting that Polignac's conjecture[6] is true, the same author has extended Green & Tao theorem and stated that, among the primes, there always exit free n-uplets for $n > 2$. Furthermore, some simple constructive properties have been described allowing to exibit many n-uplets (up to $n = 13$).

Whenever, for a given n, there is a free n-uplet (that is to say, not necessarily with consecutive prime numbers), there are several of them and one can conjecture that there are infinitely many.

Coming back to our octuplet (or 8-uplet) in (1), one claims that there are other ones and conjecture there are infinitely many but, *a priori*, neither necessarily with strictly consecutive primes, nor with the same g (as in (2)).

We propose below a constructive description of octuplets which represent a certain anlogy with (1) : we will distinguish 4 situations by answering 4 questions.

A question of notation: We caracterize an octuplet $\mathbb{O}_8$ by its first term p_1 and its template (or pattern, or frame) g (that is the sequence of 7 successive gaps $p_i - p_1, i = 2, \ldots, 8$), whence the notation $\mathbb{O}_8(p_1, g)$; let us call $\mathbb{S}_8(p_1, g)$ the sum of its terms. For example, for the octuplet (1), $p_1 = 233$, $g = (6, 8, 18, 24, 30, 36, 38)$ and $\mathbb{S}(p_1, g) = 2024$. The specific octuplet in (1) is denoted $\mathbb{O}_8(233, g_0)$.

It is clear that for octuplets of strictly consecutive primes, the sum $\mathbb{S}(p_1, g)$ is determined by the first prime p_1.

[5] For the n-th prime number p_n, the so-called primorial $p_n\#$ is defined as the product of the first n primes:

$$p_n\# = p_1 \times p_2 \times \ldots \times p_n.$$

For example, $p_4\# = 2 \times 3 \times 5 \times 7 = 210$.

[6] Polignac's conjecture (see [11]) states that, for any even number n, there are infinitely many cases of two consecutive prime numbers with difference n.

2. The main addressed questions

Here are the 4 questions addressed in this note:

1. What are the octuplets of strictly consecutive primes with any template g whose sums are closest to 2024?

2. Are there octuplets with the same template g_0 as $\mathbb{O}_8(233, g_0)$ but made up of primes which are not necessarily consecutive?

3. Are there octuplets with the same template g_0 as $\mathbb{O}_8(233, g_0)$ and made up of strictly consecutive primes?

4. Are there octuplets of prime numbers, not necessarily strictly consecutive, not with the same template g, but with a sum $\mathbb{S}(p_1, g) = 2024$?

The numerical illustrations proposed were obtained with the software PARI/GP[7], particularly well-adapted for our computations.

Answer to Question **1**.

The two octuplets which are closest (in terms of sums) to our $\mathbb{O}_8(233, g_0)$ with sum 2024 are:

- The octuplet

$$\mathbb{O}_8(229, g_1) = (229, 233, 239, 241, 251, 257, 263, 269), \tag{3}$$

that is with first term $p_1 = 229$ and template $g_1 = (4, 10, 12, 22, 28, 34, 40)$; its sum is 1982.

- The octuplet

$$\mathbb{O}_8(239, g_2) = (239, 241, 251, 257, 263, 269, 271, 277), \tag{4}$$

that is with first term $p_1 = 239$ and template $g_2 = (2, 12, 18, 24, 30, 32, 38)$; its sum is 2068.

Note that, with respect to $\mathbb{O}_8(233, g_0) = (233, \dots, 271)$, $\mathbb{O}_8(229, g_1)$ (resp. $\mathbb{O}_8(239, g_2)$) is obtained by just a shift towards the left (resp. towards the right) of the list of 8 primes constituting it (*i.e.*: the first term in $\mathbb{O}_8(229, g_1)$ is 229, the prime just before 233 ; the last term in $\mathbb{O}_8(239, g_2)$ is 277, the prime just after 271).

These 2 octuplets $\mathbb{O}_8(229, g_1)$ and $\mathbb{O}_8(239, g_2)$ are indeed made up of strictly consecutive primes, 6 of them where present in $\mathbb{O}_8(233, g_0)$, but with completely different templates.

[7]PARI/GP (developed at the University of Bordeaux) is a cross platform and open-source computer algebra system designed for fast computations in number theory.

Answer to Question **2**.

From the theoretical viewpoint, we know that, as soon that there exists a solution to the posed question with some template g, there are many and one even conjectures they are infinitely many. Indeed, the first terms p_1 of these octuplets with a common template g_0 can easily be calculated ; here is the list of results lying in the interval $[2, 10^{11}]$, they are 67:

$$23, 233, 26693, 2721413, 7447043, 154670903, 200559053, 213108653,$$
$$232777673, 273788363, 299267663, 334388273, 395453543, 459117353,$$
$$477709283, 527326403, 1015923113, 1563572243, 1688692763,$$
$$2426018723, 2918492243, 3226427423, 3743134523, 4445599853,$$
$$4458163943, 4697619593, 5493835013, 5546977823, 5655191873,$$
$$5930389313, 6131660663, 6470661143, 6866364353, 7598587943,$$
$$7694171333, 7730278313, 7822909943, 8671898363, 8794809893,$$
$$9669694253, 9724231463, 10282110533, 11339835173, 11993475713,$$
$$12373853543, 12511447643, 13048250063, 13172825423, 13621596683,$$
$$14133919403, 14403637733, 14647192583, 14844463853, 14950135433,$$
$$14963445443, 15917145113, 16229019473, 16298722043, 16440570533,$$
$$16465195133, 16562434373, 16614037463, 16828762463, 16881552893,$$
$$17450628323, 18320500403, 19156220393.$$

The first one, that is the one which begins with the samallest p_1 is with $p_1 = 23$, is:

$$\mathbb{O}_8(23, g_0) = (23, 29, 31, 41, 47, 53, 59, 61); \tag{5}$$

its sum is 344. Observe that the primes 37 and 43 are not present in this octuplet; therefore $\mathbb{O}_8(23, g_0)$ is not made up of strictly consecutive primes as was $\mathbb{O}_8(233, g_0)$.

$\mathbb{O}_8(233, g_0)$ is indeed the next one answering our question. The one coming just after $\mathbb{O}_8(233, g_0)$ begings with $p_1 = 26693$, it is:

$$\mathbb{O}_8(26693, g_0) = (26693, 26699, 26701, 26711, 26717, 26723, 26729, 26731); \tag{6}$$

its sum is 213704. Here again, the primes mading up $\mathbb{O}_8(26693, g_0)$ are not strictly consecutive; the prime 26713 between 26711 and 26717 is not present.

Answer to Question **3**.

The condition imposed on the successive primes p_i appearing in the octuplet $\mathbb{O}_8(p_1, g_0)$ is stronger than for those in the previous question. Here are the first terms of the octuplets answering the present question, lying in the same interval $[2, 10^{11}]$, they are 50:

$$233, 2721413, 154670903, 200559053, 232777673, 273788363,$$
$$299267663, 459117353, 527326403, 1015923113, 156357243,$$
$$1688692763, 2426018723, 2918492243, 3743134523, 4445599853,$$
$$4458163943, 4697619593, 5493835013, 5546977823, 5930389313,$$
$$6131660663, 6470661143, 7598587943, 7694171333, 7730278313,$$
$$7822909943, 8671898363, 8794809893, 9669694253, 9724231463,$$
$$10282110533, 11339835173, 12373853543, 12511447643,$$
$$13048250063, 13621596683, 14133919403, 14950135433,$$
$$14963445443, 15917145113, 16229019473, 16298722043,$$
$$16440570533, 16465195133, 16562434373, 16828762463,$$
$$16881552893, 17450628323, 19156220393.$$

We pursued our computations up to 5×10^{11}, it took about 4 hours of computing time. Here are the next results for p_1, hence following 19156220393; they are 43:

$$20414953253, 20713088153, 21870999173, 23408723543,$$
$$23461155923, 24806273543, 26848746143, 27892521953,$$
$$28237569593, 28482221063, 28545603473, 29740038533,$$
$$31261980983, 33843674993, 33908161583, 33953850443,$$
$$34051906583, 34271422943, 35165108183, 36154640573,$$
$$36515717933, 36964445723, 39289641533, 39659666993,$$
$$39874322693, 40083718523, 42451142603, 42553981703,$$
$$44027165513, 4439516063, 45053465753, 46096733993,$$
$$46117341503, 46648484213, 47685125693, 48309842723,$$
$$48316650713, 48769687913, 49121048363, 49505852153,$$
$$49838475983, 49933070483, 50766482153.$$

Two observations in the context of this Question 3:

- The octuplet $\mathbb{O}_8(223, g_0)$ is the *first one* answering the question, therefore with the *smallest first term* p_1; it is precisely *the octuplet of the year* 2024, which was the initial motivation of our thinking.

- We have to wait a long time for the second one, since $\mathbb{O}_8(2721413, g_0)$ has for sum 21771464...; hence the next year with the same described property as 2024 belongs to the 21772nd millenium!

Answer to Question **4**.

An immediate answer to this question is provided by a slight modification of the octuplet in (1): substituting 227 for 233 (as a result, -6 in the sum) and 277 for 271 (hence $+6$ in the sum). The resulting octuplet is with $g_4 = (12, 14, 24, 30, 36, 42, 50)$:

$$\mathbb{O}_8(227, g_4) = (227, 239, 241, 251, 257, 263, 269, 277). \tag{7}$$

No octuplet answering our question begins with 2, the smallest prime; the reason is that, otherwise, the sum $\mathbb{S}(2, g) = 2 + (7 \text{ odd numbers})$ would be odd. So the smallest possible first prime p_1 is 3. Indeed, an answer to the posed question, actually with this smallest $p_1 = 3$, is the following one:

$$\mathbb{O}_8(3, g_5) = (3, 5, 7, 11, 13, 17, 19, 1949). \tag{8}$$

As expected, the sum in (8) is $(3 + 5 + ... + 19) = 75 + 1949 = 2024$, indeed a funny wink to one of the authors, born in 1949, and who therefore celebrates his 75th birthday this year 2024. The proposed octuplet $\mathbb{O}_8(3, g_5) = (3, ..., 1949)$ is somehow *an extremal one*.

We have seen that the smallest possible p_1 in the aimed octuplet is 3; according to what we have exhibited in (8), the largest possible 8th prime p_8 is 1949, the 296th prime. So, since the first prime 2 has been excluded, the number of solutions to our problem is bounded above by $\begin{pmatrix} 295 \\ 8 \end{pmatrix}$. This is a rough upper bound, something like 10^{15}, which could be refined, but the point here is to note that the number of solutions to our problem is *finite*.

We have calculated the number of answers to our question in some cases. We denote by $\mathrm{nb}(p_1)$ the number of octuplets solutions which begin with p_1. For example, $\mathrm{nb}(233) = 1$ because it turns out that the only solution with $p_1 = 233$ is the octuplet $\mathbb{O}_8(233, g_0)$ considered from the beginning. Here are further results,

$$\begin{aligned} \mathrm{nb}(229) &= 6, ..., \mathrm{nb}(211) = 596, ..., \\ \mathrm{nb}(127) &= 1907469, ..., \mathrm{nb}(59) = 32983105, ... \end{aligned}$$

Here is the list of 6 octuplets solutions beginning with $p_1 = 229$:

$$229, 233, 239, 241, 251, 257, 263, 311;$$

$$229, 233, 239, 241, 251, 257, 281, 293;$$
$$229, 233, 239, 241, 257, 263, 269, 293;$$
$$229, 233, 239, 251, 257, 263, 269, 283;$$
$$229, 233, 239, 251, 257, 263, 271, 281;$$
$$229, 233, 241, 251, 257, 263, 269, 281.$$

All the computing codes used in the present work, as well as others (see [2]), can be obtained from the first author.

Final observations:

- For the first 3 questions, there are infinitely many solutions.

- On the other hand, the constraint imposed on the octuplets answering the 4th question being very strong, there are only a finite number of solutions. But, of course, this in no way invalidates either the theorem by B. Green and T. Tao ([7]) or its extension ([4]) which concern free n-uplets.

3. Conclusion and perspective

In this note, we played with the integer 2024 corresponding to the current year, in particular the starting point described in the decomposition (1). It is questionable whether a similar situation will occur in the foreseeable future. Here are some answers:

- In 2030, we will have a 10-uplet of stricly consecutive primes whose sum is 2030:

$$2030 = 179 + 181 + 191 + 193 + 197 + 199 + 211 + 223 + 227 + 229.$$

- In 2032, we will have a couple of twin primes and a quadruplet of primes summing up 2032:

$$2032 = 1013 + 1019;$$

$$2032 = 499 + 503 + 509 + 521.$$

- In 2034, it will be the turn of sextuplets:

$$2034 = 317 + 331 + 337 + 347 + 349 + 353.$$

- Finally, in 2040 we end with a 12-uplet:

$$\begin{aligned} 2040 &= 139 + 149 + 151 + 157 + 163 + 167 \\ &\quad +173 + 179 + 181 + 191 + 193 + 197. \end{aligned}$$

Later, as the saying goes, "Whoever lives will see".

References

1. C. K. Caldwell and G. L. Honaker, Jr., *Prime Curios! The Dictionary of Prime Number Trivia.* CreateSpace (2009).

2. R. L. Clerc, *Calculs mathématiques en ligne.*

Site http://sayrac.rlc.free.fr/ sur internet. Website managed by R. L. Clerc, English version available.

3. R. L. Clerc, *Arithmetic subsets and constellations of prime numbers* (In French, *Sous-ensembles et constellations arithmétiques de nombres premiers).* https://hal.archives-ouvertes.fr,hal-03589472, (2022).

4. R. L. Clerc, *Arbitrary constellations of prime numbers* (In French, *Constellations quelconques de nombres premiers*).

https://hal.archives-ouvertes.fr,hal-03606289), (2022).

5. J.-P. Delahaye, *Merveilleux nombres premiers. Voyage au coeur de l'arithmétique.* Editions Belin-Pour la Science (2000). New edition (with two additional chapters) in 2013.

6. J. Derbyshire, *Dans la jungle des nombres premiers.* Collection Quai des sciences, Dunod (2007).

7. B. Green & T. Tao, *The primes contain arbitrarily long arithmetic progressions.* Annals of Mathematics, vol. 167, p. 481 – 547 (2008).

8. J.-B. Hiriart-Urruty, *Ah ! les nombres premiers... 57 pintxos sur ces "incassables" de l'arithmétique (Ah! prime numbers... 57 pintxos on these "unbreakable" numbers in arithmetic).* Booklet to appear in 2024.

9. F. Kennard, *Unsolved problems in mathematics.* AMF Publishing (2015).

10. D. Lignon, *Dictionary of (almost) all integers* (in French, *Dictionnaire de (presque) tous les nombres entiers).* Editions Ellipses, Paris (2012).

11. A. de Polignac, *Recherches nouvelles sur les nombres premiers.* Comptes Rendus Acad. Sciences de Paris, tome 29, p. 397 – 401 (1849).

12. P. Ribenboim, *Nombres premiers : mystères et records.* Presses Universitaires de France (1994).

13. P. Ribenboim, *The new book of prime number records.* Springer-Verlag, New-York (1996).

14. M. Du Sautoy, *La symphonie des nombres premiers.* Editions Héloïse d'Ormesson (2005).